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Solution of Solvable Irreducible Quintic Equations, without the aid of a Resolvent Sextic.

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THE PROBLEM STATED.

§1. Jerrard has proved that a quintic equation can always be brought to the trinomial form $F(x) = x^5 + p_4 x + p_5 = 0$. (1)
Hence the problem of the solution of the general equation of the fifth degree is reduced to that of the solution of (1). Let $F(x)$ be irreducible. Then since the equation $F(x) = 0$ cannot be solved algebraically except in particular cases, it is incumbent on the Algebraist, first, to find a criterion of its solvability. In other words, if

$$p_4 = \phi(A, B, \text{etc.}), \text{ and } p_5 = \psi(A, B, \text{etc.}),$$

where ϕ and ψ denote rational functions of certain quantities, A, B, etc., he has to discover the most general forms of the expressions $\phi(A, B, \text{etc.})$ and $\psi(A, B, \text{etc.})$ that are compatible with the possibility of exhibiting the roots of the equation $F(x) = 0$ as algebraical functions of A, B, etc. Next, assuming that the functions ϕ and ψ are such as to render the equation $F(x) = 0$ solvable, he has to solve the equation, that is, to obtain its roots in terms of A, B, etc. These are the two things proposed to be done in the present paper.

THE CONCLUSION REACHED.

§2. We may take for granted that p_4 is distinct from zero. Then it will be found that the coefficients p_4 and p_5, in the most general forms they can receive consistent with the solvability of the equation (1), are rational functions of two quantities A and B. More definitely,

$$\left.\begin{aligned} p_4 &= \frac{5A^4(3 - B)}{16 + B^2} \\ \text{and } p_5 &= \frac{A^5(22 + B)}{16 + B^2}\,. \end{aligned}\right\} \qquad (2)$$

The relation between p_4 and p_5 indicated in (2) is *the necessary condition of the solvability of the equation* (1). Assuming now that this relation subsists, take λ, a root of the equation

$$x^4 - Bx^3 - 6x^2 + Bx + 1 = 0. \tag{3}$$

$$\left.\begin{aligned} \text{Put } a &= \frac{-(\lambda^2+1)}{A\lambda(\lambda-1)} \\ \text{and } \theta &= \frac{-A^5\lambda(\lambda-1)^2}{(16+B^2)(\lambda+1)(\lambda^2+1)}\,. \end{aligned}\right\} \tag{4}$$

Then *the root* r *of the equation* $F(x)=0$ is

$$r = \theta^{\frac{1}{5}} + a\theta^{\frac{2}{5}} + \lambda a^2\theta^{\frac{3}{5}} - \lambda a^3\theta^{\frac{4}{5}}. \tag{5}$$

CRITERION OF SOLVABILITY: THE EQUATIONS (2) SHOWN TO BE NECESSARY.

§3. I will now prove that, if the equation (1) be solvable algebraically, the coefficients must be of the forms given in (2).

§4. The root r of the equation (1) may be expressed as follows:

$$r = \theta^{\frac{1}{5}} + a\theta^{\frac{2}{5}} + b\theta^{\frac{3}{5}} + c\theta^{\frac{4}{5}}; \tag{6}$$

where a, b, c, involve only surds subordinate to $\theta^{\frac{1}{5}}$. In fact, these coefficients are rational functions of θ. Taking the second, third, fourth and fifth powers of r, and arranging according to the powers of $\theta^{\frac{1}{5}}$ lower than the fifth,

$$\left.\begin{aligned} r^2 &= d + d_1\theta^{\frac{1}{5}} + d_2\theta^{\frac{2}{5}} + d_3\theta^{\frac{3}{5}} + d_4\theta^{\frac{4}{5}} \\ r^3 &= g + g_1\theta^{\frac{1}{5}} + g_2\theta^{\frac{2}{5}} + g_3\theta^{\frac{3}{5}} + g_4\theta^{\frac{4}{5}} \\ r^4 &= h + h_1\theta^{\frac{1}{5}} + h_2\theta^{\frac{2}{5}} + h_3\theta^{\frac{3}{5}} + h_4\theta^{\frac{4}{5}} \\ r^5 &= k + k_1\theta^{\frac{1}{5}} + k_2\theta^{\frac{2}{5}} + k_3\theta^{\frac{3}{5}} + k_4\theta^{\frac{4}{5}}; \end{aligned}\right\} \tag{7}$$

where d, d_1, g, etc. are clear of $\theta^{\frac{1}{5}}$. If S_c be the sum of the c^{th} powers of the roots of the equation (1), $S_2 = 0 \therefore d = 0$; and $S_3 = 0 \therefore g = 0$. Also

$$S_4 = 5h, \text{ and } S_5 = 5k \;\therefore\; 5h = -4p_4, \text{ and } k = -p_5.$$

The value of d is $2\theta(c + ab)$. But $d = 0$. Therefore

$$c = -ab. \tag{8}$$

Again, $g = 3\theta\{(a^2 + b) + \theta(ac^2 + b^2c)\}$. But $g = 0$. Therefore

$$(a^2 + b) + \theta(ac^2 + b^2c) = 0. \tag{9}$$

Suppose, if possible, that $a = 0$. Then, by (8), $c = 0$. Therefore, by (9), $b = 0$. Hence (6) becomes $r = \theta^{\frac{1}{5}}$; and the equation $F(x) = 0$ is $x^5 - \theta = 0$. But this makes p_4 zero; which is at variance with the assumption in §2. Therefore we cannot have $a = 0$. We may consequently put $b = \lambda a^2$. Therefore, by (8), $c = -\lambda a^3$. This reduces the expression for r in (6) to the form (5); and at the same time changes equation (9) into

$$a^5\theta\lambda^2(\lambda-1)=\lambda+1. \tag{10}$$

§5. The values of d_1, d_2, d_3, d_4, are, keeping in view (10),

$$\begin{aligned} d_1 &= \theta(b^2+2ac)=a^4\theta\lambda(\lambda-2), \\ d_2 &= 1+2bc\theta=-\frac{\lambda+3}{\lambda-1}, \\ d_3 &= c^2\theta+2a=\frac{a(3\lambda-1)}{\lambda-1}, \\ d_4 &= a^2+2b \ =a^2(2\lambda+1). \end{aligned}$$

The values of g_1, g_2, g_3, g_4, are

$$\begin{aligned} g_1 &= \theta(a^3+6ab+3c)+3bc^2\theta^2 = \frac{a^3\theta(6\lambda^2+\lambda-1)}{\lambda-1}, \\ g_2 &= \theta(3a^2b+6ac+3b^2)+c^3\theta^2=\frac{\lambda a^4\theta(3\lambda^2-7\lambda+2)}{\lambda-1}, \\ g_3 &= 1+\theta(3a^2c+6bc+3ab^2) = -\frac{2\lambda^2+7\lambda+3}{\lambda(\lambda-1)}, \\ g_4 &= 3a+\theta(b^3+3c^2+6abc) = \frac{a(\lambda^2+\lambda-6)}{\lambda-1}. \end{aligned}$$

§6. It will be convenient to put

$$\begin{aligned} B &= 3-\frac{p_4(\lambda-1)}{5a\theta(\lambda+1)}, \\ f_1(x) &= x^4-3x^3-6x^2+3x+1, \\ f_2(x) &= x^4-Bx^3-6x^2+Bx+1, \\ f_3(x) &= x^4+Bx^3-6x^2-Bx+1, \\ f_4(x) &= (x^2+1)(x^4+22x^3-6x^2-22x+1), \end{aligned}$$

$$p_4^5(22+B)^4\{f_5(x)\}=p_4^5(22+B)^4(x^2+1)^4-5^5p_5^4(3-B)^5x^2(x^2-1)^2.$$

§7. By squaring the first of equations (7), and putting the result equal to the value of r^4 in the third, and again by multiplying together the first two of equations (7), and putting the result equal to the value of r^5 in the fourth,

$$h=2\theta(d_1d_4+d_2d_3)$$

and

$$k=\theta(g_1d_4+g_2d_3+g_3d_2+g_4d_1).$$

Therefore, because $5h=-4p_4$, and $k=-p_5$,

$$\begin{aligned} -4p_4 &= 10\theta(d_1d_4+d_2d_3) \\ -p_5 &= \theta(g_1d_4+g_2d_3+g_3d_2+g_4d_1). \end{aligned}$$

In these results substitute the values of d_1, g_1, etc., in §5. Then, keeping in view the forms of $f_1(x)$ and $f_4(x)$ in §6,

$$5a\theta\{f_1(\lambda)\}+p_4\lambda(\lambda-1)^2=0, \tag{11}$$

$$\theta\{f_4(\lambda)\}+p_5\lambda^2(\lambda-1)^3=0. \tag{12}$$

Hence it can be shown that

$$f_2(\lambda)=0,\ \text{and}\ f_5(\lambda)=0. \tag{13}$$

The first of equations (13) is obtained by eliminating $a\theta$ from (11) by means of the value of B in §6. To find the second, eliminate θ from (12) by means of (11) and (10). The result is

$$5^5p_5^4\lambda(\lambda^2-1)\{f_1(\lambda)\}^5+p_4^5\{f_4(\lambda)\}^4=0. \tag{14}$$

Now, because $\quad f_2(\lambda)=0,\ f_4(\lambda)=f_4(\lambda)-(\lambda^2+1)\{f_2(\lambda)\}.$

But $\quad f_4(\lambda)=(\lambda^2+1)(\lambda_4+22\lambda^3-6\lambda^2-22\lambda+1),$

and $\quad (\lambda^2+1)\{f_2(\lambda)\}=(\lambda^2+1)(\lambda^4-B\lambda^3-6\lambda^2+B\lambda+1).$

Therefore $\quad f_4(\lambda)=(\lambda^2+1)(\lambda^3-\lambda)(22+B). \qquad (15)$

Also, the equation $f_2(\lambda)=0$, otherwise written, is

$$f_1(\lambda)=-(3-B)\lambda(\lambda^2-1). \tag{16}$$

By substituting in (14) the values of $f_1(\lambda)$ and $f_4(\lambda)$ in (16) and (15),

$$(22+B)^4p_4^5(\lambda^2+1)^4-5^5p_5^4(3-B)^5\lambda^2(\lambda^2-1)^2=0.$$

By reference to §6, this will be seen to give the second of equations (13).

§8. The expression B is rational; that is to say, it is a rational function of the quantities, whatever they may be, that enter rationally into the coefficients of the equation $F(x)=0$. For, the first and fourth separate members of the value of r in (6), namely $\theta^{\frac{1}{5}}$ and $c\theta^{\frac{4}{5}}$, are respectively what are called u_1 and u_4 in an Article entitled "*Resolution of Solvable Equations of the Fifth Degree,*" that appeared in Vol. VI of this Journal. But, p_2 being the coefficient of x^3 in the equation $F(x)=0$, u_1u_4 or what is here called $c\theta$ is (Vol. VI, p. 104) the sum of $-\frac{p_2}{10}$ and a quantity which is the square root of a rational expression. And $p_2=0$. Therefore $(c\theta)^2$ or $\lambda^2a^6\theta^2$ is rational. Therefore, by (10), $\frac{(\lambda+1)a\theta}{(\lambda-1)}$ is rational. Therefore, by the value of B in §6, B is rational.

§9. Putting

$$\left.\begin{aligned} P&=p_4^5(22+B)^4 \\ Q&=5^5p_5^4(3-B)^5, \end{aligned}\right\} \tag{17}$$

and

we have, from the forms of $f_2(x)$ and $f_3(x)$ in §6,

$$P\{f_2(x)\}\{f_3(x)\}=P\{f_5(x)\}+x^2(x^2-1)^2\{Q-P(16+B^2)\}. \tag{18}$$

By (13), λ is a root of each of the equations $f_2(x)=0$ and $f_5(x)=0$. Therefore, by (18), it must be a root of the equation $x(x^2-1)=0$, unless $Q-P(16+B^2)=0$. But the only roots of the equation $x(x^2-1)=0$ are zero and $1^{\frac{1}{2}}$, neither of which is a root of the equation $f_2(x)=0$. Therefore

$$Q-P(16+B^2)=0; \tag{19}$$

or, from (17),

$$p_4^5(16+B^2)(22+B)^4=5^5p_5^4(3-B)^5. \tag{20}$$

Since, by §8, B is rational, (20) implies that $\frac{p_4(16+B^2)}{5(3-B)}$ is the fourth power of a rational quantity, say of A. Put

$$\left.\begin{aligned} \frac{p_4(16+B^2)}{5(3-B)} &= A^4, \\ \text{therefore, from (20),}\quad p_4 A(22+B) &= 5p_5(3-B). \end{aligned}\right\} \tag{21}$$

The two equations (21) give us the forms of p_4 and p_5 in (2). Hence the necessity of these forms is established.

SOLUTION OF THE EQUATION $F(x) = 0$.

§10. Having shown that p_4 and p_5 must be of the forms (2) in order that equation (1) may be solvable algebraically, I will now, taking p_4 and p_5 to be of these forms, deduce the rule given in (2) for the solution of the equation. This will prove the criterion of solvability afforded by the equations (2) to be sufficient, as it has already been seen to be necessary.

§11. Because $p_4 = \frac{5A^4(3-B)}{16+B^2}$, and $p_5 = \frac{A^5(22+B)}{16+B^2}$, the equation (19) subsists. Also, from the forms of $f_2(x)$, $f_3(x)$ and $f_5(x)$, in §6, equation (18) subsists. Therefore, from (19),

$$\{f_2(x)\}\{f_3(x)\} = f_5(x). \tag{22}$$

Let λ be a root of equation (3). Then $f_2(\lambda) = 0$. It follows exactly as in §7 that

$$\left.\begin{aligned} f_1(\lambda) &= -(3-B)\,\lambda\,(\lambda^2-1) \\ \text{and } f_4(\lambda) &= (\lambda^2+1)(\lambda^3-\lambda)(22+B) \end{aligned}\right\} \tag{23}$$

Now, from (22), because $f_2(\lambda) = 0$, $f_5(\lambda) = 0$. Therefore

$$(22+B)^4 p_4^5 (\lambda^2+1)^4 - 5^5 p_5^4 (3-B)^5 \lambda^2 (\lambda^2-1)^2 = 0.$$

Substitute here the values of $22+B$ and $3-B$ in (23). The result is equation (14). Take a and θ as in (4). Then

$$a\theta = \frac{A^4(\lambda-1)}{(16+B^2)(\lambda+1)}.$$

By the first of equations (2), this is equivalent to

$$a\theta = \frac{p_4(\lambda-1)}{5(3-B)(\lambda+1)}.$$

And this again, when we substitute for $3-B$ its value in (23), becomes equation (11).

From the second of equations (4) eliminate $16+B^2$ by means of the second of equations (2). Then $$\theta = \frac{-p_5\lambda(\lambda-1)^2}{(22+B)(\lambda+1)(\lambda^2+1)};$$

and this, when we substitute for $22 + B$ its value in (23), becomes equation (12). Eliminate p_4 and p_5 from (14) by means of (11) and (12). The result is equation (10).

§12. Give r the value it has in (5). Let

$$r_1,\ r_2,\ r_3,\ r_4,\ r_5, \tag{24}$$

be the five values of r obtained by writing for $\theta^{\frac{1}{5}}$ successively the five expressions $\theta^{\frac{1}{5}}$, $w\theta^{\frac{1}{5}}$, $w^2\theta^{\frac{1}{5}}$, $w^3\theta^{\frac{1}{5}}$, $w^4\theta^{\frac{1}{5}}$, w being a primitive fifth root of unity. Let S'_c be the sum of the c^{th} powers of the terms in (24), while S_c is the sum of the c^{th} powers of the roots of equation (1). Then it can be shown that

$$S_1' = S_1,\ S_2' = S_2,\ S_3' = S_3,\ S_4' = S_4,\ S_5' = S_5. \tag{25}$$

For, S_1' and S_2' are both identically zero. This makes $S_1' = S_1$, and $S_2' = S_2$. Also

$$S_3' = 15a^2\theta\{(\lambda + 1) - a^5\theta\lambda^2(\lambda - 1)\}.$$

Therefore, from (10), $S_3' = 0 = S_3$. Again, if d_1, d_2, g_1, etc., be taken as in §5, namely $d_1 = a^4\theta\lambda(\lambda - 2)$, and so on,

$$S_4' = 10\theta(d_1d_4 + d_2d_3)$$

$$\text{and } S_5' = 5\theta(g_1d_4 + g_2d_3 + g_3d_2 + g_4d_1).$$

Therefore, as in §7,

$$\left.\begin{aligned} S_4' &= \frac{20a\theta\{f_1(\lambda)\}}{\lambda(\lambda-1)^2} \\ \text{and } S_5' &= \frac{5\theta\{f_4(\lambda)\}}{\lambda^2(\lambda-1)^3}. \end{aligned}\right] \tag{26}$$

Now $S_4 = -4p_4$, and $S_5 = -5p_5$. Therefore, from (26) compared with (11) and (12), $S_4' = S_4$, and $S_5' = S_5$. Thus all the equations in (25) are established.

§13. Let $X = 0$ be the equation whose roots are the terms in (24). Then

$$X = x^5 - \tfrac{1}{4}S_4'x - \tfrac{1}{5}S_5'.$$

Hence, because $S_4' = S_4$, and $S_5' = 5$,

$$X = x^5 - \tfrac{1}{4}S_4x - \tfrac{1}{5}S_5 = x^5 + p_4x + p_5 = F(x).$$

This makes r a root of the equation $F(x) = 0$.

VERIFYING INSTANCES.

§14. As A and B may have any values whatever, let $A = 5$, and $B = 2$. Then, by (2), equation (1) is

$$x^5 + \frac{625}{4}x + 3750 = 0.$$

The equation (3), for determining λ, is

$$x^4 - 2x^3 - 6x^2 + 2x + 1 = 0.$$

A root of this quartic is $\lambda = 3.52$. Hence, by (4),

$$\theta = -57.71, \text{ and } a = -.3019.$$

Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -2.250, \\ a\theta^{\frac{2}{5}} &= -1.529, \\ \lambda a^2\theta^{\frac{3}{5}} &= -3.656, \\ -\lambda a^3\theta^{\frac{4}{5}} &= 2.484. \end{aligned}$$

Therefore

$$r = \theta^{\frac{1}{5}} + a\theta^{\frac{2}{5}} + \lambda a^2\theta^{\frac{3}{5}} - \lambda a^3\theta^{\frac{4}{5}} = -4.951.$$

§15. Second example. Let $A = 1$, $B = -1$. Then equation (1) is

$$x^5 + \frac{20x}{17} + \frac{21}{17} = 0.$$

The equation for determining λ is

$$x^4 + x^3 - 6x^2 - x + 1 = 0.$$

A root of this quartic is $\lambda = 2.0496$. Hence

$$\theta = -.00837, \text{ and } a = -2.4176.$$

Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -.384, \\ a\theta^{\frac{2}{5}} &= -.357, \\ \lambda a^2\theta^{\frac{3}{5}} &= -.679, \\ -\lambda a^3\theta^{\frac{4}{5}} &= .631, \\ r &= -.789. \end{aligned}$$

§16. Third example. Let $A = 1$, $B = 0$. Then equation (1) is

$$x^5 + \frac{15x}{16} + \frac{11}{8} = 0.$$

The equation for determining λ is

$$x^4 - 6x^2 + 1 = 0.$$

A root of this quartic is $\lambda = 2.414215$. Hence

$$\theta = -.01294, \quad a = -2.$$

Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -.419194, \\ a\theta^{\frac{2}{5}} &= -.351447, \\ \lambda a^2\theta^{\frac{3}{5}} &= -.711345, \\ -\lambda a^3\theta^{\frac{4}{5}} &= .596384, \\ r &= -.8856. \end{aligned}$$

§17. Fourth example. Let $A = 1$, $B = 7$. Then equation (1) is

$$x^5 - \frac{4x}{13} + \frac{29}{65} = 0.$$

The equation for determining λ is

$$x^4 - 7x^3 - 6x^2 + 7x + 1 = 0.$$

A root of this quartic is $\lambda = .7690975$. Hence

$$\theta = -.0002246, \text{ and } a = 8.9619.$$

Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -.186239, \\ a\theta^{\frac{2}{5}} &= .310846, \\ \lambda a^2\theta^{\frac{3}{5}} &= -.399025, \\ -\lambda a^3\theta^{\frac{4}{5}} &= -.665999, \\ r &= -.940617. \end{aligned}$$

§18. Fifth example. Let $A = 1$, $B = -7$. Then equation (1) is

$$x^5 + \frac{10x}{13} + \frac{3}{13} = 0.$$

The equation for determining λ is

$$x^4 + 7x^3 - 6x^2 - 7x + 1 = 0.$$

A root of this quartic is $\lambda = 1.300226$. Hence

$$\theta = -.00029133, \quad a = -6.89253.$$

Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -1.963, \\ a\theta^{\frac{2}{5}} &= -.2655, \\ \lambda a^2\theta^{\frac{3}{5}} &= -.4671, \\ -\lambda a^3\theta^{\frac{4}{5}} &= .6319, \\ r &= -.297. \end{aligned}$$

§19. Sixth example. Let $A = 1$, and $B = \sqrt{2}$. Then equation (1) is

$$18x^5 + 5(3 - \sqrt{2})x + (22 + \sqrt{2}) = 0.$$

The equation for determining λ is

$$\left(x^2 - \frac{\sqrt{2}}{2}x - 1\right)^2 = \frac{9x^2}{2}.$$

A root of this quartic is $\lambda = \sqrt{2} + \sqrt{3}$. Therefore

$$\begin{aligned} \theta^{\frac{1}{5}} &= -.4469, \\ a\theta^{\frac{2}{5}} &= -.3223, \\ \lambda a^2\theta^{\frac{3}{5}} &= -.7313, \\ -\lambda a^3\theta^{\frac{4}{5}} &= .5275, \\ r &= -.973. \end{aligned}$$